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LETTER TO THE EDITOR

Parafermions and surface exponents of self-dual $Z(N)$ spin models from conformal invariance

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Abstract. We study the critical behaviour of a family of self-dual two-dimensional spin models with $Z(N)$ symmetry ($N \leq 8$) in its Hamiltonian formulation. Using the relations between the mass-gap amplitudes of the Hamiltonian in a finite strip and the critical exponents we estimate the anomalous dimensions of the parafermions occurring in the underlying field theory as well as the surface exponents.

In the past few years much attention has been devoted to the study of two-dimensional $Z(N)$ spin systems, firstly because they are non-trivial generalisations of the Ising model (Domany and Riedel 1979, Alcaraz and Köberle 1980, Cardy 1980) and secondly because of their similarity to four-dimensional $Z(N)$ gauge systems (Fradkin and Susskind 1978, Elitzur *et al* 1979, Kogut 1979, Creutz *et al* 1979, Alcaraz and Köberle 1981). These spin models are self-dual and for $N \geq 5$ they exhibit a soft phase which is the precursor of the disordered Gaussian-like phase of the planar XY model.

Fateev and Zamolodchikov (1982) obtained a special family of $Z(N)$ models which is exactly soluble at its self-dual point. The free energy was obtained by solving the associated star-triangle equations (Baxter 1982). The one-dimensional quantum Hamiltonian H_N associated with these models was derived recently (Alcaraz and Lima Santos 1986) and is given by

$$H_N = -\sum_{k=1}^{L} \sum_{n=1}^{N-1} \{[S^n(k)S^{+n}(k+1) + R^n(k)]/\sin(\pi n/N)\} \tag{1}$$

where $S(k)$ and $R(k)$ are $N \times N$ matrices defined at the lattice sites $1 \leq k \leq L$ and obey the $Z(N)$ algebra

$$[S(k), R(l)] = [S(k), S(l)] = [R(k), R(l)] = 0 \qquad k \neq l$$
$$S(k)R(k) = \exp(i2\pi/N)R(k)S(k) \qquad R^N(k) = S^N(k) = 1.$$

Zamolodchikov and Fateev (1985) constructed a set of $(1+1)$-dimensional field theories with the properties of $Z(N)$ invariance and self-duality like the above statistical models. These theories are the natural candidates for the underlying field theories governing the critical behaviour of the above statistical models. These theories are conformally invariant with their central charge, or conformal anomaly, of their Virasoro algebra given by

$$C = 2(N-1)/(N+2). \tag{2}$$

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From their conformal algebra the predicted values of the scaling dimensions associated with the $(N-1)$-order or disordered ($Z(N)$ charged) operators of the statistical model are

$$2d_n = n(N-n)/N(N+2) \qquad n = 1, 2, \ldots, N-1 \tag{3}$$

while the scaling dimensions associated with the $\bar{N}$-thermal ($Z(N)$ neutral) operators are

$$2D_n = 2n(n+1)/(N+2) \qquad n = 1, 2, \ldots, \bar{N} \tag{4}$$

where $\bar{N}$ is the integer part of $N/2$. These theories also predicted the existence of parafermions with spin s and anomalous dimension $X_{\mathrm{Pf}}(s, n)$ given by

$$S = l(N-n-l)/N \qquad X_{\mathrm{Pf}}(s, n) = 2d_n + s \qquad l = 0, 1, 2, \ldots, N-n \tag{5}$$

and

$$s = l(N-l)/N \qquad X_{\mathrm{Pf}}(s, n) = 2d_n + s \tag{6}$$

where $d_n (n = 1, 2, \ldots, N-1)$ is given by (3). For $N = 2, 3$ these models correspond to the critical Ising and three-state Potts models, for $N = 4$ it is a particular point of the quantum Ashkin–Teller model (Kohmoto *et al* 1981) and for $N > 4$ the above dimensions correspond to the exponents of the antiferromagnetic critical point of the RSOS model (Andrews *et al* 1984, Huse 1984).

In previous publications (Alcaraz 1986, 1987) by calculating the spectrum of the associated Hamiltonian (1) in a finite lattice with periodic boundary conditions and exploring the consequences of the conformal symmetry of the infinite critical system (Cardy 1986b) we verified with good precision the predictions 2.1–2.3 for $N \leqslant 8$. In this letter the mass-gap amplitudes will be used in order to calculate the anomalous dimensions of the parafermions occurring in the model as well the several possible surface exponents associated with (1). We shall consider the Hamiltonian (1) with $\tilde{n}$-twisted boundary conditions

$$S(L+1) = \exp(\mathrm{i}2\pi\tilde{n}/N)S(1) \qquad \tilde{n} = 0, 1, 2, \ldots, N-1$$

$\tilde{n} = 0$ corresponding to the periodic case, and free boundary conditions, $S(L+1) = 0$, imposed. The Hamiltonian (1) commutes with the $Z(N)$-charge operator

$$\exp(\mathrm{i}2\pi Q/N) = \prod_{k=1}^{L} R(k)$$

and consequently in the R basis it can be block separated N-disjoint sectors labelled by $Q = n = 0, 1, 2, \ldots, N-1$. In the case of twisted boundary conditions, these blocks can be further block diagonalised according to their momentum†, while in the case of free ends these sectors can be further block separated in two sectors according to the parity under reflexion of the lattice. In the following let us denote by $E_n^{(\tilde{n})}(k, r)$ $(r = 0, 1, 2, \ldots)$ the r-excited state with momentum k in the sector $Q = n$ of the Hamiltonian (1) with $\tilde{n}$-twisted boundary conditions imposed. In the case of free ends we denote by $E_n^{(\mathrm{E})}(r, p)$ $(r = 0, 1, 2, \ldots; p = \pm)$ the r-excited state with parity p in the sector with charge $Q = n$.

† See, for example, von Gehlen and Rittenberg (1986) for a proper definition of momentum in the case $\tilde{n} \neq 0$.

The conformal invariance at criticality of the infinite statistical model produces many important implications in two dimensions (see Cardy 1986b for a review). Specifically, Cardy (1984a, 1986a, b) has derived a set of important relations between the mass-gap amplitudes of the transfer matrix (or associated Hamiltonian) of the statistical system in a finite strip and the anomalous dimension of the operators describing the critical behaviour of the infinite system. Let us initially concentrate on the evaluation of the scaling dimensions $X_n^{(\tilde{n})}$ of the parafermionic operator (Fradkin and Kadanoff 1980) with spin $n\tilde{n}/N$. These dimensions, in the Hamiltonian formalism, can be obtained by extrapolating ($L \to \infty$) the sequence (Cardy 1986b, von Gehlen *et al* 1986)

$$X_n^{(\tilde{n})}(L) \equiv L[E_n^{(\tilde{n})}(0,0) - E_0^{(0)}(0,0)]/2\pi\zeta \qquad n, \tilde{n} = 1, 2, \ldots, N-1. \tag{7}$$

The constant ζ is unity in the transfer matrix (Euclidean) formalism but is model dependent for Hamiltonians (Alcaraz and Drugowich de Felício 1984, von Gehlen *et al* 1986). This constant can be extracted in several ways, for example by extrapolating the sequences

$$\zeta(L) = L[E_n^{(0)}(2\pi/L, 0) - E_n^{(0)}(0,0)]/2\pi \qquad n \neq 0. \tag{8}$$

For the Hamiltonian (1) previous numerical analysis (Alcaraz 1986, 1987) indicates the conjecture that $\zeta = N$. This value will be assumed hereafter.

In table 1 we present for $Z(5)$, $Z(6)$, $Z(7)$ and $Z(8)$ the extrapolated values of the sequences (7) corresponding to several parafermions occurring in the model described by (1). All the numerical calculations of eigenenergies in this letter were performed by using the Lanczos method (Roomany *et al* 1980, Hamer and Barber 1981a, b). The extrapolated values quoted in table 1 were obtained by using the alternate ε algorithm (Hamer and Barber 1981b) for lattice sizes up to $L=9$, 8, 7 and 7 for $N=5$, 6, 7 and

Table 1. Extrapolated and conjectured results for the scaling dimensions $X_n^{(\tilde{n})}$ of the parafermions with spin $n\tilde{n}/N$ for the $Z(N)$ ($N=5$-8) systems. The conjectured values denoted by * and † are $(N-1)/N$ and $\frac{6}{5}$, respectively, and the remaining ones are given by (4) and (5).

		$Z(5)$	$Z(6)$	$Z(7)$	$Z(8)$
$X_1^{(1)}$	Extrapolated	0.372 ± 0.001	0.335 ± 0.002	0.304 ± 0.003	0.278 ± 0.005
	Conjectured	0.371 428 ...	0.333 333 ...	0.301 587 ...	0.275
$X_2^{(1)}$	Extrapolated	0.5713 ± 0.0003	0.5209 ± 0.0003	0.479 ± 0.001	0.440 ± 0.002
	Conjectured	0.571 428 ...	0.520 833 ...	0.476 190 ...	0.4375
$X_3^{(1)}$	Extrapolated	0.7144 ± 0.0004	0.6660 ± 0.0005	0.619 ± 0.001	0.575 ± 0.001
	Conjectured	0.714 285 ...	0.666 666 ...	0.619 047 ...	0.575
$X_4^{(1)}$	Extrapolated	0.8001 ± 0.0004	0.7703 ± 0.0005	0.729 ± 0.002	0.6875 ± 0.0005
	Conjectured	0.8*	0.770 833 ...	0.730 158 ...	0.6875
$X_5^{(1)}$	Extrapolated	—	0.835 ± 0.004	0.805 ± 0.005	0.774 ± 0.003
	Conjectured	—	0.8333 333 ...*	0.809 523 ...	0.775
$X_6^{(1)}$	Extrapolated	—	—	0.858 ± 0.002	0.83 ± 0.01
	Conjectured	—	—	0.857 142 ...*	0.8375
$X_7^{(1)}$	Extrapolated	—	—	—	0.876 ± 0.005
	Conjectured	—	—	—	0.875*
$X_2^{(2)}$	Extrapolated	0.914 ± 0.001	0.833 ± 0.002	0.761 ± 0.002	0.703 ± 0.005
	Conjectured	0.914 285 ...	0.833 333 ...	0.761 904 ...	0.7
$X_3^{(2)}$	Extrapolated	1.200 ± 0.002	—	1.01 ± 0.02	0.94 ± 0.01
	Conjectured	1.2†	—	1.015 873 ...	0.9375

8, respectively. The conjectured values in table 1 not marked with * or † are given by (5) and (6). We can clearly see that agreement with the conjectured values is very good, which once more (Alcaraz 1986, 1987, Jimbo *et al* 1986) indicates that the $Z(N)$ field theory of Zamolodchikov and Fateev (1985) is the underlying field theory describing the criticality of these statistical systems. For $N=2$, 3 and 4 the relations (2)-(5) were verified analytically and numerically (von Gehlen *et al* 1986, Alcaraz *et al* 1987). Apart from parafermions with dimensions predicted by (4) and (5) our results also indicate the presence of parafermions with spin $(N-1)/N$ and dimension $X_1^{(N-1|)}=(N-1)/N$. These scaling dimensions are obtained by generating the zero-momentum state of the Hamiltonian (1) in the sector $n=1$ and with $\tilde{n}=(N-1)$-twisted boundary conditions imposed. In table 1 we present these results together with the conjectured values denoted by *. Moreover, for the particular case of the $Z(5)$ model, another non-predicted parafermion denoted by † in table 1 with spin $\frac{6}{5}$ and dimension $X_3^{(2)}=\frac{6}{5}$ was also obtained.

The surface exponents which govern the various correlations along the surface of the semi-infinite two-dimensional model can also be obtained by exploiting the conformal invariance of the infinite system. By conformally transforming a given a statistical model in the half-plane into a strip of size L a set of important relations have been derived (Cardy 1984a). In the Hamiltonian formalism these relations can be stated as follows. Corresponding to each surface exponent X_s of the infinite system (Binder 1983, Cardy 1986b) there exists, in the Hamiltonian with size L and free boundaries, a set of states with eigenenergies at the bulk critical point given by

$$E_s(r)=E_0(L)+\pi\zeta(X_s+r)/L+O(L^{-1}) \qquad r=0,1,2,\ldots. \tag{9}$$

The constant $\zeta=N$ is the same as that occurring in (7) and $E_0(L)$ is the ground-state energy of the finite chain. From the above relation we can identify each sector of the Hamiltonian (1) with free ends with a surface exponent $X_s^{(n,p)}$ by extrapolating the sequences

$$X_s^{(n,0)}\equiv[E_n^{(F)}(0,p)-E_0^{(F)}(0,+)]L/\pi\zeta \qquad n=1,2,\ldots,N-1;\ p=\pm \tag{10}$$

$$X_s^{(0,+)}\equiv[E_0^{(F)}(1,+)-E_0^{(F)}(0,+)]L/\pi\zeta \tag{11}$$

$$X_s^{(0,-1)}\equiv[E_0^{(F)}(0,-1)-E_0^{(F)}(0,+)]L/\pi\zeta \tag{12}$$

where $E_0^{(F)}$ $(0,+)$ is the ground state of the finite chain. In table 2 we present the extrapolated values of these sequences for the $Z(5)$, $Z(6)$, $Z(7)$ and $Z(8)$ Hamiltonians by using lattice sizes up to $L=8$, 7, 7 and 6 respectively. We observe from these results that $X_s^{(n,-)}(\infty)=X_s^{(n,+)}(\infty)+1$, implying that, for each charge n, the ground state in the positive (negative) parity sector corresponds to the first (second) state in the tower of states given by (9). Our numerical results indicate that, like the $N=2$, 3 and 4 cases (Cardy 1984b), the surface exponent $X_s^{(e)}=n_{\parallel}^{(e)}/2$ corresponding to the energy operator is given by

$$X_s^{(e)}\equiv X_s^{(0,+)}(\infty)=2 \tag{13}$$

and the surface exponents corresponding to the two-point correlations of the $(N-1)$th-order parameters $X_s^{(n)}=\eta_{\parallel}^{(n)}/2$ are given by

$$X_s^{(n)}\equiv X_s^{(n,+)}(\infty)=\frac{4d_n}{D_n}=\frac{n(N-n)}{N} \qquad n=1,2,\ldots,N-1. \tag{14}$$

Table 2. Extrapolated and conjectured results for the surface exponents $X_s^{(n,p)}$ for the $Z(N)$ ($N=5$–8) systems. The conjectured values are given by (13) and (14).

		$Z(5)$	$Z(6)$	$Z(7)$	$Z(8)$
$X_s^{(0,+)}$	Extrapolated	2.000 ± 0.001	2.00 ± 0.01	2.00 ± 0.02	2.01 ± 0.03
	Conjectured	2.0	2.0	2.0	2.0
$X_s^{(0,-)}$	Extrapolated	3.00 ± 0.02	2.99 ± 0.01	3.00 ± 0.03	3.00 ± 0.03
	Conjectured	3.0	3.0	3.0	3.0
$X_s^{(1,+)}$	Extrapolated	0.797 ± 0.004	0.83 ± 0.01	0.85 ± 0.01	0.86 ± 0.01
	Conjectured	0.8	0.8333 . . .	0.8571 . . .	0.875
$X_s^{(1,-)}$	Extrapolated	1.798 ± 0.002	1.83 ± 0.01	1.85 ± 0.01	1.86 ± 0.03
	Conjectured	1.8	1.8333 . . .	1.8571 . . .	1.875
$X_s^{(2,+)}$	Extrapolated	1.203 ± 0.004	1.33 ± 0.01	1.42 ± 0.01	1.50 ± 0.03
	Conjectured	1.2	1.3333 . . .	1.4285 . . .	1.5
$X_s^{(2,-)}$	Extrapolated	2.206 ± 0.002	2.35 ± 0.02	2.45 ± 0.03	2.58 ± 0.05
	Conjectured	2.2	2.3333 . . .	2.4285 . . .	2.5
$X_s^{(3,+)}$	Extrapolated	1.203 ± 0.004	1.50 ± 0.01	1.71 ± 0.01	1.88 ± 0.03
	Conjectured	1.2	1.5	1.7142 . . .	1.875
$X_s^{(3,-)}$	Extrapolated	2.206 ± 0.002	2.55 ± 0.04	2.78 ± 0.05	—
	Conjectured	2.2	2.5	2.7142 . . .	2.875
$X_s^{(4,+)}$	Extrapolated	0.797 ± 0.04	1.33 ± 0.01	1.71 ± 0.01	1.99 ± 0.03
	Conjectured	0.8	1.3333 . . .	1.7142 . . .	2.0

The conjectured values quoted in table 2 are those predicted by (13) and (14). It is interesting to observe that (14) is a particular case of the more general relation $X_s^{(o)} = 4X_v^{(o)}/X_v^{(e)}$ between the surface exponent $X_s^{(o)}$ of an order parameter, its scaling dimension $X_v^{(o)}$ and the scaling dimension $X_v^{(e)}$ ($n=1$ in (3)) of the energy operator in the bulk system. This last relation can be verified for the $q \leqslant 4$ state Potts, Ashkin-Teller and O(N) models.

In summary, by exploiting the implications of the conformal invariance of the statistical system at criticality we have calculated, for the family of Hamiltonians (1) ($N=5$–8), the scaling dimensions of the parafermions and the surface exponents. Beyond the family of parafermions predicted by the $Z(N)$ field theory of Zamolodchikov and Fateev (1985) (table 1) we have also obtained another family of parafermionic operators with spin and dimension $(N-1)/N$. Our numerical results for the surface exponents (table 2) strongly suggest the conjectures (13) and (14).

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References

Alcaraz F C 1986 *J. Phys. A: Math. Gen.* **19** L1085–92
—— 1987 *J. Phys. A: Math. Gen.* **20** 2511–26
Alcaraz F C, Barber M N and Batchelor M T 1987 *Phys. Rev. Lett.* **58** 771–4
Alcaraz F C and Drugowich de Felício J R 1984 *J. Phys. A: Math. Gen.* **17** L651–5
Alcaraz F C and Köberle R 1980 *J. Phys. A: Math. Gen.* **13** L153–60
—— 1981 *J. Phys. A: Math. Gen.* **14** 1169–92
Alcaraz F C and Lima Santos A 1986 *Nucl. Phys.* B **275** [FS17] 436–58

Andrews G F, Baxter R J and Forrester P J 1984 *J. Stat. Phys.* **35** 193-266
Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
Binder K 1983 *Phase Transitions and Critical Phenomena* vol 8, ed C Domb and J L Lebowitz (New York: Academic) p 1
Cardy J L 1980 *J. Phys. A: Math. Gen.* **13** 1507-15
—— 1984a *J. Phys. A: Math. Gen.* **17** L385
—— 1984b *Nucl. Phys.* B **240** [FS12] 514-32
—— 1986a *Nucl. Phys.* B **270** [FS16] 186
—— 1986b *Phase Transitions and Critical Phenomena* vol 11, ed C Domb and J L Lebowitz (New York: Academic)
Creutz M, Jacobs L and Rebbi C 1979 *Phys. Rev.* D **20** 1915
Domany E and Riedel E K 1979 *Phys. Rev.* B **19** 5817
Elitzur S, Pearson R B and Shigemitsu J 1979 *Phys. Rev.* D **19** 3698
Fateev V A and Zamolodchikov A B 1982 *Phys. Lett.* **92A** 37-9
Fradkin E and Kadanoff L P 1980 *Nucl. Phys.* B **170** 1
Fradkin E and Susskind L 1978 *Phys. Rev.* D **17** 2637-58
Hamer C J and Barber M N 1981a *J. Phys. A: Math. Gen.* **14** 259-74
—— 1981b *J. Phys. A: Math. Gen.* **14** 2009-25
Huse D A 1984 *Phys. Rev.* B **30** 3908
Jimbo M, Miwa T and Okado M 1986 *Nucl. Phys.* B **275** [FS17] 517-45
Kogut J B 1979 *Rev. Mod. Phys.* **51** 659
Kohmoto M, den Nijs M and Kadanoff L P 1981 *Phys. Rev.* B **24** 5229-41
Roomany H H, Wild H W and Holloway L E 1980 *Phys. Rev.* D **21** 1557
von Gehlen G and Rittenberg V 1986 *J. Phys. A: Math. Gen.* **19** L625
von Gehlen G, Rittenberg V and Ruegg H 1986 *J. Phys. A: Math. Gen.* **19** 107-19
Zamolodchikov A B and Fateev V A 1985 *Zh. Eksp. Teor. Fiz.* **89** 380-99 (1985 *Sov. Phys.-JETP* **62** 215-25)